Maximum Flow Problem

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Course:Algorithms on Graphs

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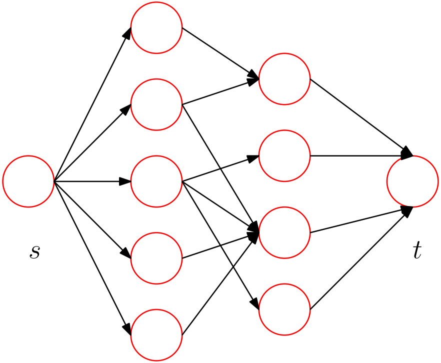
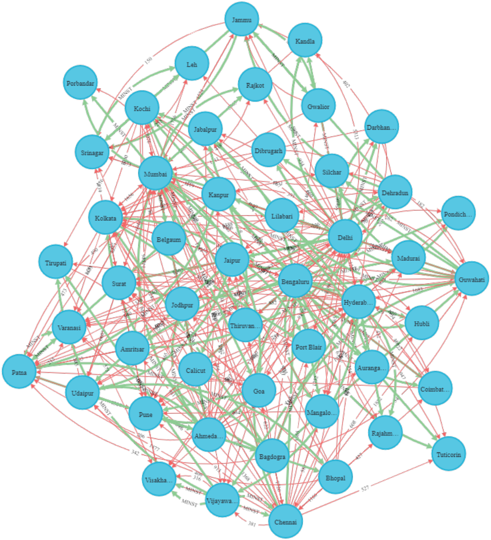


How many edge-disjoint paths are there from s to t?



Can you answer the same question in this graph?

Let us further generalize this problem.

Flow Network

Flow Network

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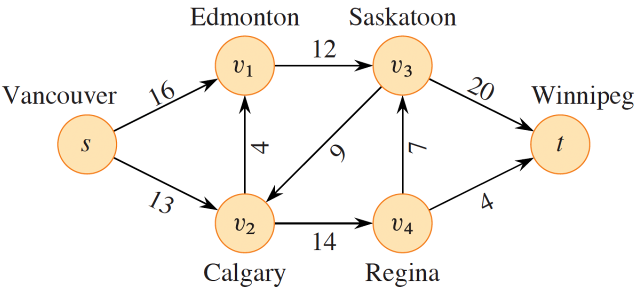
- If E contains an edge (u, v), then there is no edge (v, u) in the reverse direction.
- If $(u, v) \notin E$, then for convenience we define $c(u, v) = 0$, and we disallow self-loops.



An example of a flow network.

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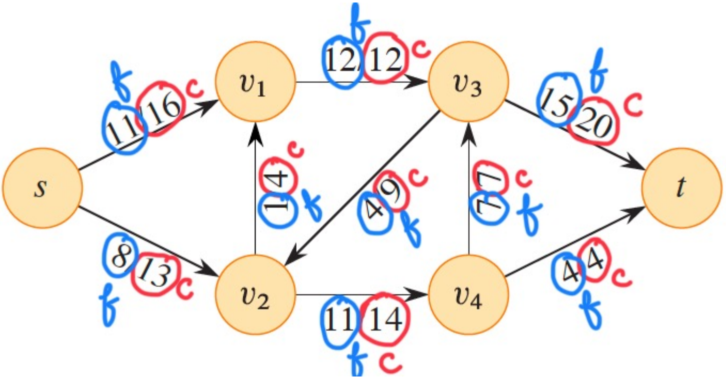
- **Capacity constraint:** For all $u, v \in V$, we require

$$0 \leq f(u, v) \leq c(u, v).$$

- **Flow conservation:** For all $u \in V \setminus \{s, t\}$, we require

$$\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v).$$

  The total flow in must equal the total flow out.



An example of flow in a flow network. Entries on edges represent flow and capacity values.

Value of the Flow

The value $|f|$ of a flow f is defined as

$$|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s).$$

That is, the total flow out of the source minus the flow into the source.

Maximum-flow Problem

Input: A flow network G with source s and sink t.
Goal: To find a flow of maximum value.

Why is this an interesting problem?

Can model many problems, like:

- liquids flowing through pipes
- parts through assembly lines
- current through electrical networks
- information through communication networks

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Applications in:

- Electrical Power Transmission
- Airline Scheduling
- Communication Networks
- in finding Bipartite Matching
- Robustness (in case an edge fails)

The Ford-Fulkerson Method

The Ford-Fulkerson Method

We call it a “method” rather than an “algorithm” because it encompasses several implementations with differing running times.

Three Important Components:

- Residual Networks.
- Augmenting Paths.
- Cuts.

Brief Idea:

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- The flow is updated using these paths until no more augmenting paths exist.
- The max-flow min-cut theorem guarantees this process yields the maximum flow.

Residual Capacity

We define the residual capacity $c_f(u, v)$ for a pair of vertices $u, v \in V$ by

$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v) & \text{if } (u, v) \in E, \\ f(v, u) & \text{if } (v, u) \in E, \\ 0 & \text{otherwise.} \end{cases} \qquad (\star)$$

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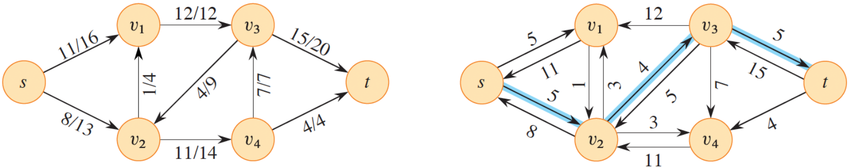
Note: 1. Exactly one case of $(\star)$ applies to each ordered pair of vertices.

2. We define residual capacity corresponding to some fixed flow f.

Residual Network

Given a flow network $G = (V, E)$ and a flow f, the residual network of G induced by f is $G_f = (V, E_f)$, where

$$E_f = \{(u, v) \in V \times V \mid c_f(u, v) > 0\}.$$


s
v_1
v_2
v_3
v_4
t
11/16
12/12
15/20
8/13
1/4
4/9
7/7
11/14
4/4
s
v_1
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5
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4

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- Residual Networks. ✓
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Augmenting Path

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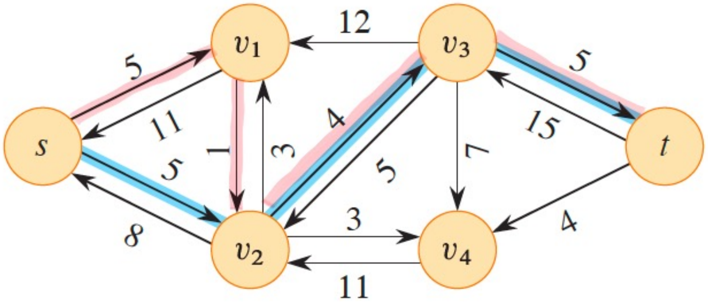

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Lemma: $G = (V, E)$: flow network
f: a flow in G
p : an augmenting path in G_f.
Define a function $f_p\colon V \times V \to \mathbb{R}$ by

$$f_p(u, v) = \begin{cases} c_f(p) & \text{if } (u, v) \text{ is on } p, \\ 0 & \text{otherwise.} \end{cases}$$

Then, f_p is a flow in G_f with value $|f_p| = c_f(p) > 0$.

Flow Augmentation

f : flow in G
f' : flow in the corresponding residual network G_f
$f \uparrow f' : V \times V \to \mathbb{R}$, such that

$$(f \uparrow f')(u, v) = \begin{cases} f(u, v) + f'(u, v) - f'(v, u) & \text{if } (u, v) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma: Let $G = (V, E)$ be a flow network with source s and sink t, and let f be a flow in G. Let G_f be the residual network of G induced by f, and let f' be a flow in G_f. Then:

1. $f \uparrow f'$ is a flow in G.
2. value of $|f \uparrow f'| = |f| + |f'|$.

The Ford-Fulkerson method repeatedly augments the flow along augmenting paths until it has found a maximum flow.

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How do we know that when the algorithm terminates, it has actually found a maximum flow?

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What is a Cut in a flow network?

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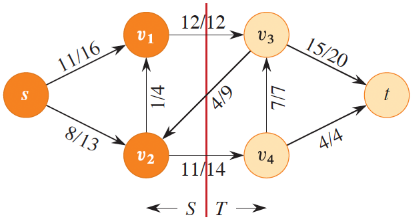

If f is a flow, then the net flow $f(S, T)$ across the cut (S, T) is:

$$f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u, v) - \sum_{u \in S} \sum_{v \in T} f(v, u).$$

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The capacity of the cut (S, T) is

$$c(S, T) = \sum_{u \in S} \sum_{v \in T} c(u, v).$$

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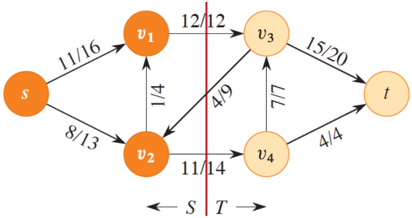

A minimum cut of a network is a cut whose capacity is minimum over all cuts of the network.

Lemma: Let f be a flow in a flow network G with source s and sink t, and let (S, T) be any cut of G. Then the net flow across (S, T) is $f(S, T) = |f|$.

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Proof.

Long cumbersome proof. Just use definitions! Can you try it yourself later, please? $\square$

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Proof.

30 seconds proof! $\square$

Max-flow min-cut theorem

If f is a flow in a flow network $G = (V, E)$ with source s and sink t, then the following conditions are equivalent:

1. f is a maximum flow in G.
2. The residual network G_f contains no augmenting paths.
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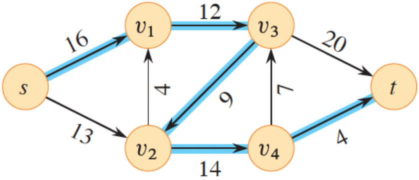
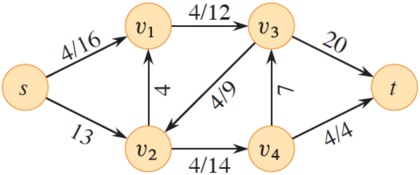
Proof.

I think VV's can prove easily on the spot! $\square$

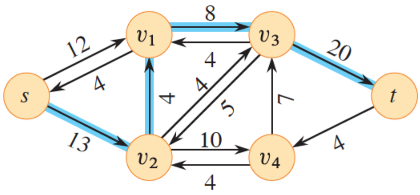
Step 1



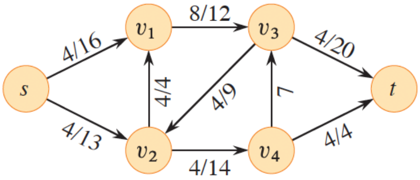
Step 2



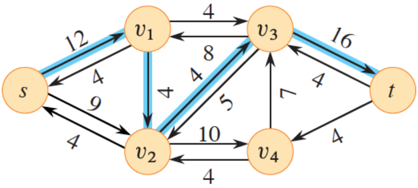
Step 3



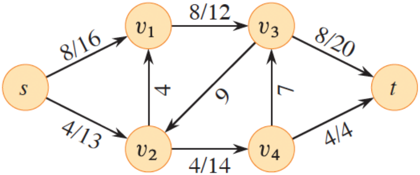
Step 4



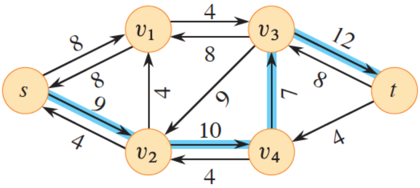
Step 5



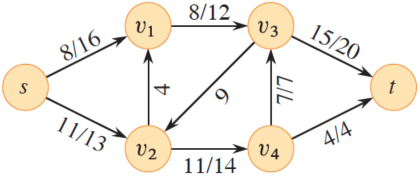
Step 6



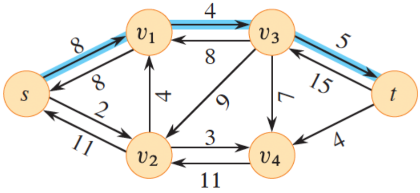
Step 7



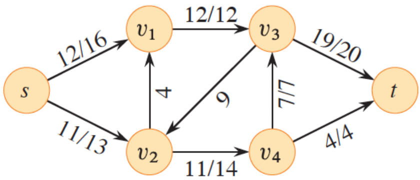
Step 8



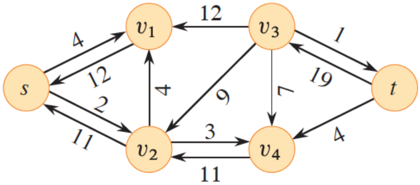
Step 9



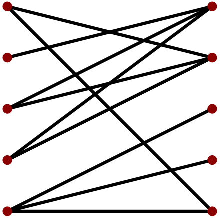
Step 10



Step 11

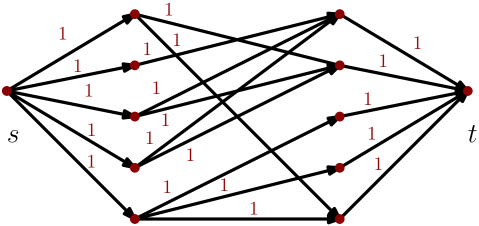


Applications



Find maximum matching in this bipartite graph using Ford-Fulkerson.

You will need the following construction and lemma.



Lemma

If the capacity function c takes only integer values, then $f(u, v)$ is an integer for all (u, v).

Thank You.